

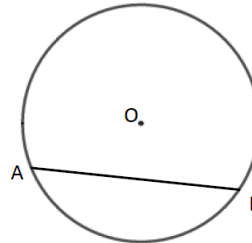
Circle theorems

A LEVEL LINKS

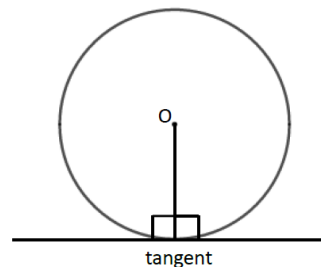
Scheme of work: 2b. Circles – equation of a circle, geometric problems on a grid

Key points

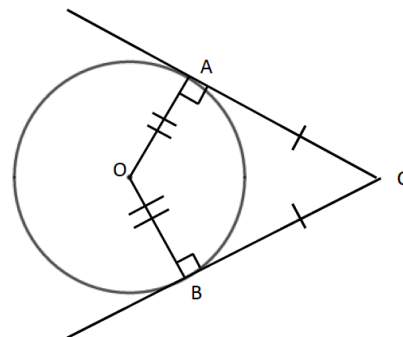
- A chord is a straight line joining two points on the circumference of a circle.
So AB is a chord.



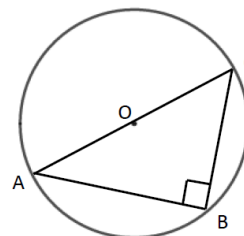
- A tangent is a straight line that touches the circumference of a circle at only one point.
The angle between a tangent and the radius is 90° .



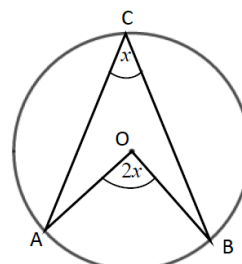
- Two tangents on a circle that meet at a point outside the circle are equal in length.
So $AC = BC$.



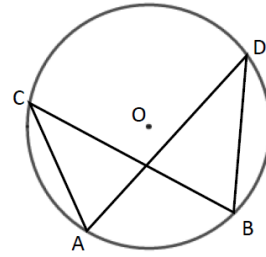
- The angle in a semicircle is a right angle.
So angle $ABC = 90^\circ$.



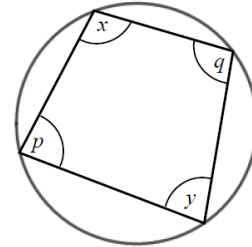
- When two angles are subtended by the same arc, the angle at the centre of a circle is twice the angle at the circumference.
So angle $AOB = 2 \times$ angle ACB .



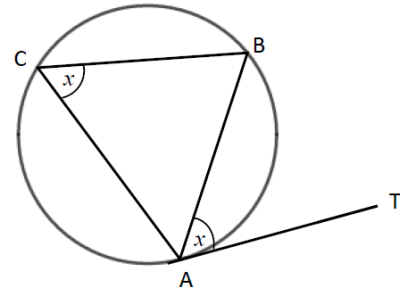
- Angles subtended by the same arc at the circumference are equal. This means that angles in the same segment are equal.
So angle $ACB = \text{angle } ADB$ and angle $CAD = \text{angle } CBD$.



- A cyclic quadrilateral is a quadrilateral with all four vertices on the circumference of a circle. Opposite angles in a cyclic quadrilateral total 180° .
So $x + y = 180^\circ$ and $p + q = 180^\circ$.

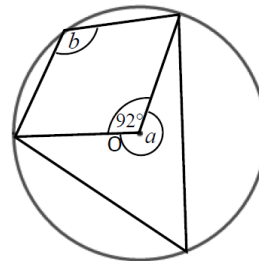


- The angle between a tangent and chord is equal to the angle in the alternate segment, this is known as the alternate segment theorem.
So angle $BAT = \text{angle } ACB$.



Examples

- Example 1** Work out the size of each angle marked with a letter.
Give reasons for your answers.

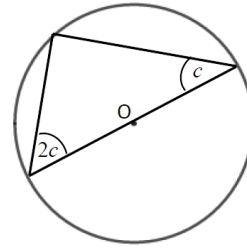


Angle $a = 360^\circ - 92^\circ$
 $= 268^\circ$
 as the angles in a full turn total 360° .

Angle $b = 268^\circ \div 2$
 $= 134^\circ$
 as when two angles are subtended by the same arc, the angle at the centre of a circle is twice the angle at the circumference.

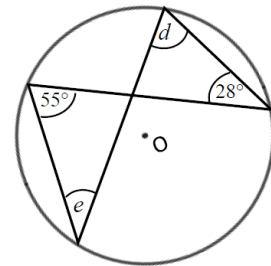
- The angles in a full turn total 360° .
- Angles a and b are subtended by the same arc, so angle b is half of angle a .

Example 2 Work out the size of the angles in the triangle.
Give reasons for your answers.



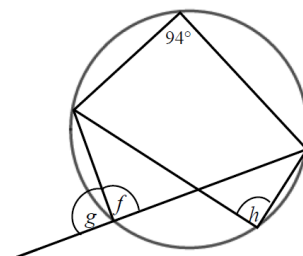
<p>Angles are 90°, $2c$ and c.</p> $90^\circ + 2c + c = 180^\circ$ $90^\circ + 3c = 180^\circ$ $3c = 90^\circ$ $c = 30^\circ$ $2c = 60^\circ$ <p>The angles are 30°, 60° and 90° as the angle in a semi-circle is a right angle and the angles in a triangle total 180°.</p>	<ol style="list-style-type: none"> 1 The angle in a semicircle is a right angle. 2 Angles in a triangle total 180°. 3 Simplify and solve the equation.
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Example 3 Work out the size of each angle marked with a letter.
Give reasons for your answers.



<p>Angle $d = 55^\circ$ as angles subtended by the same arc are equal.</p> <p>Angle $e = 28^\circ$ as angles subtended by the same arc are equal.</p>	<ol style="list-style-type: none"> 1 Angles subtended by the same arc are equal so angle 55° and angle d are equal. 2 Angles subtended by the same arc are equal so angle 28° and angle e are equal.
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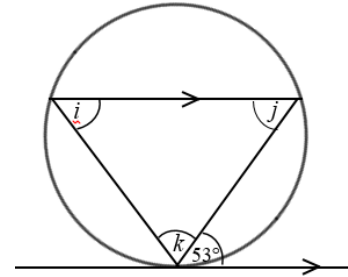
Example 4 Work out the size of each angle marked with a letter.
Give reasons for your answers.



<p>Angle $f = 180^\circ - 94^\circ$ $= 86^\circ$ as opposite angles in a cyclic quadrilateral total 180°.</p>	<ol style="list-style-type: none"> 1 Opposite angles in a cyclic quadrilateral total 180° so angle 94° and angle f total 180°. <p style="text-align: right;"><i>(continued on next page)</i></p>
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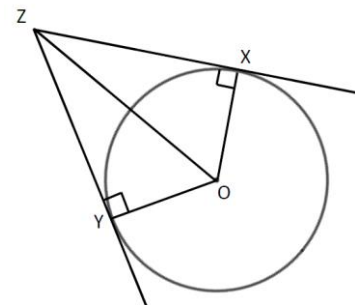
<p>Angle $g = 180^\circ - 86^\circ$ $= 84^\circ$ as angles on a straight line total 180°.</p> <p>Angle $h = \text{angle } f = 86^\circ$ as angles subtended by the same arc are equal.</p>	<p>2 Angles on a straight line total 180° so angle f and angle g total 180°.</p> <p>3 Angles subtended by the same arc are equal so angle f and angle h are equal.</p>
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Example 5 Work out the size of each angle marked with a letter. Give reasons for your answers.



<p>Angle $i = 53^\circ$ because of the alternate segment theorem.</p> <p>Angle $j = 53^\circ$ because it is the alternate angle to 53°.</p> <p>Angle $k = 180^\circ - 53^\circ - 53^\circ$ $= 74^\circ$ as angles in a triangle total 180°.</p>	<p>1 The angle between a tangent and chord is equal to the angle in the alternate segment.</p> <p>2 As there are two parallel lines, angle 53° is equal to angle j because they are alternate angles.</p> <p>3 The angles in a triangle total 180°, so $i + j + k = 180^\circ$.</p>
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Example 6 XZ and YZ are two tangents to a circle with centre O. Prove that triangles XZO and YZO are congruent.

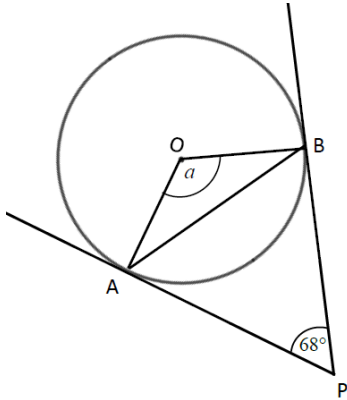


<p>Angle $OXZ = 90^\circ$ and angle $OYZ = 90^\circ$ as the angles in a semicircle are right angles.</p> <p>OZ is a common line and is the hypotenuse in both triangles.</p> <p>$OX = OY$ as they are radii of the same circle.</p> <p>So triangles XZO and YZO are congruent, RHS.</p>	<p>For two triangles to be congruent you need to show one of the following.</p> <ul style="list-style-type: none"> • All three corresponding sides are equal (SSS). • Two corresponding sides and the included angle are equal (SAS). • One side and two corresponding angles are equal (ASA). • A right angle, hypotenuse and a shorter side are equal (RHS).
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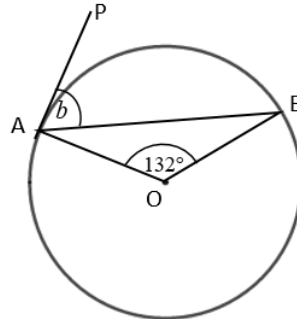
Practice

- 1 Work out the size of each angle marked with a letter.
Give reasons for your answers.

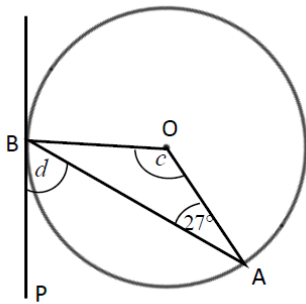
a



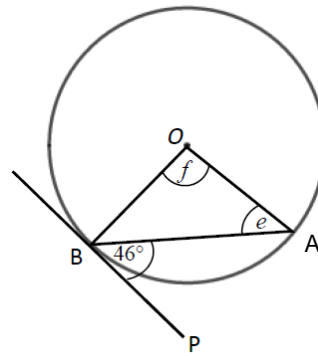
b



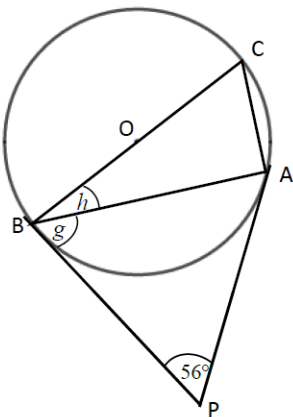
c



d

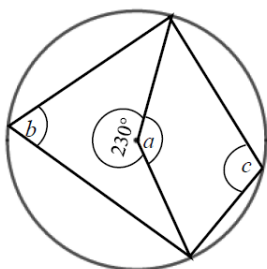


e

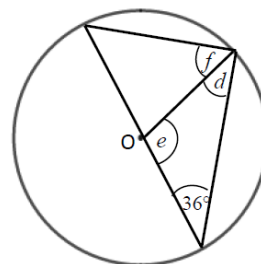


- 2 Work out the size of each angle marked with a letter.
Give reasons for your answers.

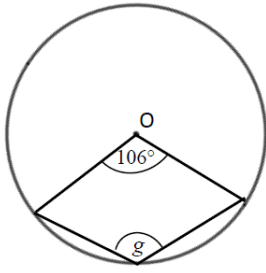
a



b



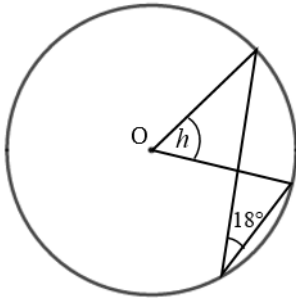
c



Hint

The reflex angle at point O and angle g are subtended by the same arc. So the reflex angle is twice the size of angle g .

d

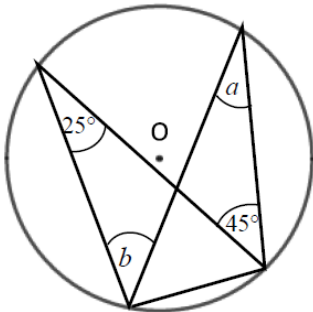


Hint

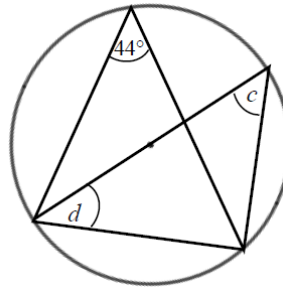
Angle 18° and angle h are subtended by the same arc.

3 Work out the size of each angle marked with a letter. Give reasons for your answers.

a



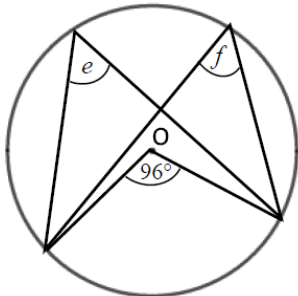
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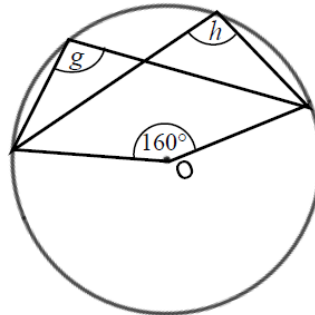
Hint

One of the angles is in a semicircle.

c

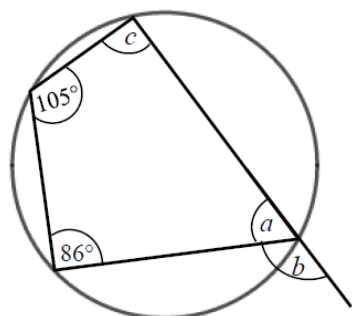


d



- 4 Work out the size of each angle marked with a letter.
Give reasons for your answers.

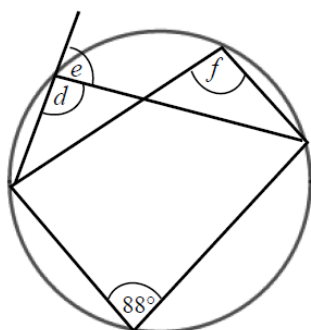
a



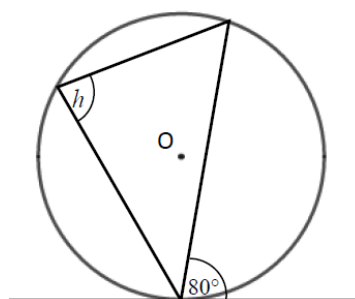
Hint

An exterior angle of a cyclic quadrilateral is equal to the opposite interior angle.

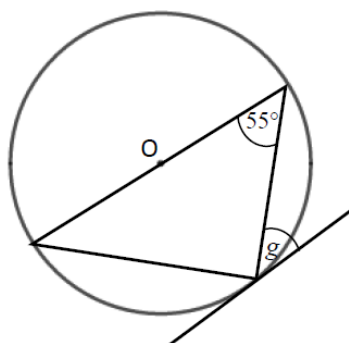
b



c



d



Hint

One of the angles is in a semicircle.

Extend

- 5 Prove the alternate segment theorem.

Answers

- 1**
- a** $a = 112^\circ$, angle $OAP = \text{angle } OBP = 90^\circ$ and angles in a quadrilateral total 360° .
 - b** $b = 66^\circ$, triangle OAB is isosceles, Angle $OAP = 90^\circ$ as AP is tangent to the circle.
 - c** $c = 126^\circ$, triangle OAB is isosceles.
 $d = 63^\circ$, Angle $OBP = 90^\circ$ as BP is tangent to the circle.
 - d** $e = 44^\circ$, the triangle is isosceles, so angles e and angle OBA are equal. The angle $OBP = 90^\circ$ as BP is tangent to the circle.
 $f = 92^\circ$, the triangle is isosceles.
 - e** $g = 62^\circ$, triangle ABP is isosceles as AP and BP are both tangents to the circle.
 $h = 28^\circ$, the angle $OBP = 90^\circ$.
- 2**
- a** $a = 130^\circ$, angles in a full turn total 360° .
 $b = 65^\circ$, the angle at the centre of a circle is twice the angle at the circumference.
 $c = 115^\circ$, opposite angles in a cyclic quadrilateral total 180° .
 - b** $d = 36^\circ$, isosceles triangle.
 $e = 108^\circ$, angles in a triangle total 180° .
 $f = 54^\circ$, angle in a semicircle is 90° .
 - c** $g = 127^\circ$, angles at a full turn total 360° , the angle at the centre of a circle is twice the angle at the circumference.
 - d** $h = 36^\circ$, the angle at the centre of a circle is twice the angle at the circumference.
- 3**
- a** $a = 25^\circ$, angles in the same segment are equal.
 $b = 45^\circ$, angles in the same segment are equal.
 - b** $c = 44^\circ$, angles in the same segment are equal.
 $d = 46^\circ$, the angle in a semicircle is 90° and the angles in a triangle total 180° .
 - c** $e = 48^\circ$, the angle at the centre of a circle is twice the angle at the circumference.
 $f = 48^\circ$, angles in the same segment are equal.
 - d** $g = 100^\circ$, angles at a full turn total 360° , the angle at the centre of a circle is twice the angle at the circumference.
 $h = 100^\circ$, angles in the same segment are equal.
- 4**
- a** $a = 75^\circ$, opposite angles in a cyclic quadrilateral total 180° .
 $b = 105^\circ$, angles on a straight line total 180° .
 $c = 94^\circ$, opposite angles in a cyclic quadrilateral total 180° .
 - b** $d = 92^\circ$, opposite angles in a cyclic quadrilateral total 180° .
 $e = 88^\circ$, angles on a straight line total 180° .
 $f = 92^\circ$, angles in the same segment are equal.
 - c** $h = 80^\circ$, alternate segment theorem.
 - d** $g = 35^\circ$, alternate segment theorem and the angle in a semicircle is 90° .

5 Angle $BAT = x$.

Angle $OAB = 90^\circ - x$ because the angle between the tangent and the radius is 90° .

$OA = OB$ because radii are equal.

Angle $OAB = \text{angle } OBA$ because the base of isosceles triangles are equal.

Angle $AOB = 180^\circ - (90^\circ - x) - (90^\circ - x) = 2x$ because angles in a triangle total 180° .

Angle $ACB = 2x \div 2 = x$ because the angle at the centre is twice the angle at the circumference.

